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Spectroscopy of gauge theories based on exceptional Lie groups

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Abstract

We generate by computer a basis of invariants for the fundamental representations of the exceptional Lie groups E_6 and E_7, up to degree 18. We discuss the relevance of this calculation for the study of supersymmetric gauge theories, and revisit the self-dual exceptional models. We study the chiral ring of G_2 to degree 13, as well as a few classical groups. The homological dimension of a ring is a natural estimator of its complexity and provides a guideline for identifying theories that have a good chance to be amenable to a solution.

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1. Introduction

In the past few years, much progress has been made in understanding the behavior of supersymmetric theories at low energy. In the best of cases, weakly coupled dual descriptions have been found for strongly coupled supersymmetric theories, which is tantamount to an exact solution of the theory in the very low energy regime and for very large distances[1].

However, many key questions have not been answered. For example, given a gauge group and matter content superfields in some representation of the gauge group, what is the low-energy behavior of the theory? For starters, most theories are free in the low-energy regime since they are not asymptotically free. Of those that are asymptotically free, most live in an interacting non-Abelian Coulomb phase. A smaller fraction have extended supersymmetry or live in a confining phase or an Abelian Coulomb phase at low energies, and are rather well understood. But the behavior of the majority of theories in a non-Abelian Coulomb phase

[1] For practical reasons, we will arbitrarily limit our context to $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions, although our results have a more general range of applicability and although there has been considerable work relevant to the issues addressed in this paper coming from string theory, extended supersymmetry, or theories in more or fewer dimensions.

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remains to be understood. Although not of much direct physical interest, it is an important mathematical physics problem.

An important insight into the behavior of theories in a non-Abelian Coulomb phase is duality [1]. It is unclear whether duality is generic or a feature of a few especially simple theories. Thus a great deal of work was done and has led to finding many more examples of duality beyond the examples of [1] for the classical groups SU, SO and Sp with matter fields in copies of the fundamental representations. The simplest examples of duality arise when the theory confines and the low-energy dual description just consists of gauge invariant mesons and baryons [2–22] and there is a claim that the theories in a confining phase have all been found and studied, at least for simple gauge groups [15, 16]. When the dual theory is not confining, there are numerous examples for the classical groups with a variety of features: terms in the electric superpotential, tensor products of groups, (anti)-symmetric or adjoint representations, finite theories, [23–42], while for the exceptional groups, there is a family of examples based on G_2 and the *Spin* groups [43–47] and isolated examples of a so-called 'self-duality' [48–52].

The battery of tests that known dualities pass are also the tools being used to search for new examples. Several of these tests require detailed knowledge of the theory being studied. This is the case for matching the flat directions, matching the spectra or the chiral rings, or for checking 't Hooft's anomaly matching conditions [53]. For all these tests, a knowledge of the gauge invariant chiral superfields is necessary.

Among the many impediments in finding more examples, one problem that is quite tractable is obtaining the structure of the chiral rings for theories of interest. In this paper, we extend the amount of data known about such gauge-invariant superfields for some specific cases. We focus our effort on the exceptional simple Lie groups E_6 and E_7 with matter quarks in many copies of the fundamental representation. We also illustrate with other groups how very detailed information can be obtained, systematically and for any representation of any Lie group, with enough computer power. In the process, we make tools, which are not new, more easily available, for generating this kind of data.

Now a short summary of the contents. In section 2, we provide a brief mathematical guide and references. In section 3, we mention the existence of two theories which must have a dual description, which offers some motivation for the calculations performed in this paper. In section 4, we find an (almost) complete list of the invariants of E_6 for copies of the 27-dimensional representation, a list of 20 invariants up to degree 18. In section 5, we show how this effort stalls for E_7, by exhibiting several hundred invariants, with many more yet to be found. The present author sees this complexity as a reasonable apology for failing to find the dual descriptions. We then move on to simple examples and recover some known mathematical results. In section 6.1, we study in some detail the syzygy chain for $SU(2)$ with fundamentals; in section 6.2, we recover the well known results for G_2; in sections 6.3–6.5, we find some invariants for copies the symmetric tensor representations of $SU(3)$, $SU(4)$ and $SU(5)$, and observe that the complexity dramatically increases with the rank of the group. We include appendices on constraints and glueballs in E_6, as well as some computer code for the LiE software that we used to do these calculations. In the conclusion, we mention some obvious extensions to this work, and briefly discuss its applicability.

2. Mathematical preliminaries

Our goal in this paper is to find a minimal list of the 'fundamental' polynomial invariants[2]. This is known as a Hilbert basis: a finite collection of invariants $I_1, \ldots, I_m$ forms a Hilbert

[2] For a section of the mathematical literature relevant to invariant theory: [55–69].

basis if every other invariant can be written as a polynomial function of the basis invariants: $I = P(I_1, \ldots, I_m)$ ([57], p 39). An important theorem of Hilbert showed that any finite system of homogeneous polynomials admits a Hilbert basis. The elements of the Hilbert basis are said to be (polynomially) independent.

There are other, less stringent, notions of independence: the invariants can be rationally, algebraically, or functionally independent. For the purposes of duality for supersymmetric theories in a non-Abelian Coulomb phase, which notion of independence is relevant? At the most basic level of matching flat directions between the electric and the magnetic theories, all that matters is functional independence: that the moduli spaces of vacuum states have the same dimensions and that the theories remain dual along the flat directions. At the level of 't Hooft anomaly matching, functional independence is clearly not enough and perhaps algebraic independence is what one is asking for: if 't Hooft's anomaly matchings are satisfied for a basic set of invariants, they would no longer be satisfied for the invariants obtained by acting with some functions on these invariants. Finally, the best one can impose from the requirement of supersymmetry is a complete isomorphism of the chiral rings [70–72]. For other theoretical work relevant to these issues, see [73–78].

The chiral rings that arise in supersymmetric gauge theories can be very simple or extremely complicated. In ring theory, there is a natural notion of the complexity of a ring, measured by its homological dimension. The invariants typically satisfy constraints, called first-order syzygies. These first-order syzygies themselves satisfy constraints, called second-order syzygies, and so forth. For the rings that we are concerned with here, Hilbert's theorem applies, and this chain of syzygies must terminate. The length of this chain is known as the homological dimension.

When the ring is freely generated, the homological dimension is zero. However, when it is not freely generated, it is typically very large. That these rings are very complicated is known by mathematicians. For example, for *irreducible* representations other than the fundamental or the adjoint, the homological dimension of E_7 is known to be larger than 26 334 (p 11 of [60])!, with similar surprisingly large lower bounds on the homological dimensions for other exceptional and spin groups. However, in this paper, we are mostly concerned with multiple copies of the fundamental representation. We will find that in these cases too, the homological dimension is likely to be very large.

3. Theories that must have a dual

Although this complexity was known to some physicists (cf p 2 of [70]), it had escaped the author. Most supersymmetric gauge theories exist in a non-Abelian Coulomb phase at long distances. Among them, just a few have the noteworthy feature that a gauge invariant baryon in their spectrum has an R-charge that is less than $2/3$. Such theories must have a dual description [1], because the spectrum of the electric theory is not in a unitary representation of the superconformal algebra. Furthermore, this baryon of charge less than $2/3$ must appear as a fundamental free field in the dual description. It is also to be expected, if the known examples can be a guide, that the full dual description will be free. Below, we give examples of such theories that must have duals. Our analysis of these examples is inconclusive, but might possibly be of interest to someone seeking to find duals.

3.1. E_6 with five flavours

One such example [52, 54] is the theory with $\mathcal{N} = 1$ supersymmetry and with gauge group E_6 and five flavours in the 27-dimensional fundamental representation. The global anomaly-free

symmetry is $SU(5) \times U(1)_R$. In the table below, we list the quantum numbers of the fields.

	E_6	$SU(5)$	$U(1)_R$
Q	27	$\square$	1/5
I_1	1	$\square\square\square$	3/5
I_2	1	$\overline{\square\square}$	6/5
I_3	1	$\square\square$	9/5
I_4	1	$\overline{\square\square\square}$	12/5

We have included the spectrum of polynomially independent gauge invariant baryons. This theory is in a non-Abelian Coulomb phase, since $SU(3)$ along its flat direction is. And because the R charge of I_1 is less than $2/3$, this theory must have a dual.

For future reference, we give the contributions of the various fields to the anomalies

Gauge	Field	$SU(5)$	$U(1)_R$	$SU(5)^3$	$SU(5)^2U(1)_R$	$U(1)_R$	$U(1)_R^3$
27	Q	$\square$	1/5	27	−108/5	−108	−1728/25
78	λ	1	1	0	0	78	$78 = 1950/25$
	Total			27	−108/5	−30	222/25
	I_1	$\square\square\square$	3/5	44	−56/5	−14	−56/25
	I_2	$\overline{\square\square}$	6/5	−15	42/5	10	2/5
	I_3	$\square\square$	9/5	15	168/5	40	128/5
	I_4	$\overline{\square\square\square}$	12/5	−44	196/5	49	2401/25
	$W_\alpha^2 Q^3$	$\square\square\square$	13/5	44	224/5	56	3584/25
	$W_\alpha^2 Q^3$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	13/5	16	176/5	64	4096/25
	$W_\alpha^2 Q^3$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	13/5	−1	16	64	1024/25
	$W_\alpha^2 Q^6$	$\square$	16/5	1	11/5	11	1331/25

We will now make a few observations, encountered during the search for a dual. To begin, we will make the hypothesis that there does exist a free dual description. This implies that the $SU(5)$ global symmetry is explicitly realized and that I_1 is the only gauge singlet which is an elementary field in the dual. Thus, the dual description found for Spin(8) in [46,52] will not be helpful here. In particular, this dual description contains several gauge-singlets transforming as 5-dimensional 4-index symmetric tensors of their global $SU(2)$. These singlets cannot all come from the free baryon I_1, but must come from I_2 or I_3. This means that their example, although an impressive accomplishment by itself, has not been 'fully' dualized.

One key problem in narrowing the search for a dual for E_6 with five flavours is understanding how to generate the baryons I_2, I_3, I_4 without generating undesirable invariants at the same time. Another clue is the matching of the 't Hooft anomaly for $SU(5)^3$. The 27 coming from the electric theory is badly matched by the 44 coming from the baryon I_1. Where do the -17 come from? -17 here is a rather large number in this context. One could imagine a decomposition $17 = 2 + 3 \cdot 5 = 5 + 3 \cdot 4$. This would mean three gauge groups in the dual, with fields transforming under $G_1 \times G_2 \times G_3 \times SU(5)$ as $(2, 1, 1, \bar{5}) \oplus (1, 3, 5, \bar{5})$ or as $(5, 1, 1, \bar{5}) \oplus (1, 3, 4, \bar{5})$. Such examples generate lots of unwanted invariants. To make progress, one would have to develop an understanding of how a superpotential can remove such unwanted invariants.

One simple way to avoid generating lots of unwanted invariants is to have I_2, I_3 as fundamental fields in the dual, but transforming under a $U(1)$ gauge symmetry. This is not necessarily an unattractive possibility, since free dual descriptions are typically not asymptotically free. However, this provides little help in cancelling the $SU(5)^3$ anomaly. One can wonder if the techniques of [80] could be of more general applicability and allow us to get information on what the free dual quarks could be, or at least what the rank or dimension of the dual gauge group is. Perhaps string theory constructions could also shed light on this problem.

Another curious feature is that the invariants of $SU(3)$ with 5 symmetric tensors are very similar to the invariants of E_6 with 5 fundamentals: namely, I_1, I_2 and I_3 are identical, while I_4 differs. The constraints among these invariants, and the chiral rings, and the glueballs, are of course different. It is not clear whether this observation has any significance.

3.2. Spin(16) with one spinor and two vectors

This is another example of an asymptotically free theory which is in a non-Abelian Coulomb phase at long distances, and which must have a dual description. Spin(16) has two real 128-dimensional spinor representations of opposite chiralities. Let us consider the theory with just one spinor $Q = 128_+$, and with N vectors $V = 16$. There is a choice of the R-charge for which the R-charge of Q and V are the same, and equal to $1 - \frac{7}{8+N}$. This is appropriate, since both Q^2 and V^2 are invariants.

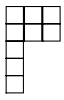
For $N = 0$, the theory confines [16], with the spectrum of invariants Q^2, Q^8, Q^{12}, Q^{14}, Q^{18}, Q^{20}, Q^{24}, Q^{30}, as follows from knowledge of the second Casimir invariants of E_8. We do not consider the case $N = 1$ here. For $N = 2$, along the Spin(14) flat direction, Q breaks into $64+64'$. One of the 64 then Higgses Spin(14) to $G_2 \times G_2$ [16]. The remaining 64-dimensional spinor presumably decomposes into $(7, 7) \oplus (7, 1) \oplus (1, 7) \oplus (1, 1)$, and these G_2 theories are known to be in a non-Abelian Coulomb phase. Since Q^2 and V^2 have R-charge equal to $3/5$, which is less than $2/3$, the Spin(16) theory with $N = 2$ must have a dual description. The difficulties in finding a dual for this theory are clearly of a very different nature than for E_6 and we will have nothing further to say about it here.

4. Invariants of E_6 with fundamentals

Our main result in this paper is a list of the polynomially independent invariants of multiple copies of the fundamental representations of E_6 and E_7, up to degree 18. We find it convenient to express our results in the context of supersymmetric gauge theories, even though they have a more general range of application. For E_6 with chiral superfields in the 27-dimensional fundamental representation, three invariants were previously well known, at least in the gauge theory community:

$I_1 = (\mathbf{27})^3_{[3]} =$ $\quad I_2 = (\mathbf{27})^6_{[2^3]} =$ $\quad I_4 = (\mathbf{27})^{12}_{[3^4]} =$.

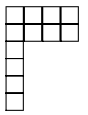
However, an invariant of lower degree than I_4, of degree 9, had attracted little attention:

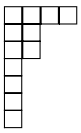
$I_3 = (\mathbf{27})^9_{[1^5\,2^2]} =$  .

Previously, it had been unclear whether

should be included as an independent invariant of E_6. We find here that E_6 does not have such an invariant. It is, however, an independent invariant of the symmetric tensor of $SU(3)$, as stated below in the section on $SU(3)$. We found two more invariants of degree 12:

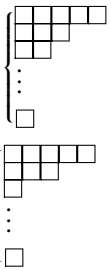
$$I_5 = (\mathbf{27})^{12}_{[1^6\,3^2]} = \quad I_6 = (\mathbf{27})^{12}_{[1^7\,1^3\,2]} = \quad .$$

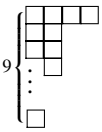


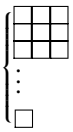


And in degree 15, we found five more invariants:

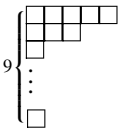
$$I_7 = (\mathbf{27})^{15}_{[1^8\,1^3\,1^2\,2]} = \quad I_8 = (\mathbf{27})^{15}_{[1^9\,1^4\,2]} = \quad I_9 = (\mathbf{27})^{15}_{[1^9\,2^3]} =$$

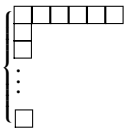






$$I_{10} = (\mathbf{27})^{15}_{[1^9\,2^2\,2]} = \quad I_{11} = (\mathbf{27})^{15}_{[1^{10}\,5]} = \quad .$$



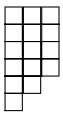


There were no constraints among the invariants up to degree 12, but five constraints (first-order syzygies) arise in degree 15:

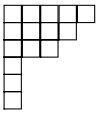
$$C_1^{(15)} = (I_1 I_4 + I_2 I_3 = 0)_{[2^5\,1^4\,1]} =$$

$$C_2^{(15)} = \left(I_1 I_2^2 + I_1^2 I_3 + I_1 I_4 + I_2 I_3 = 0\right)_{[1^5\,2^4\,2]} =$$

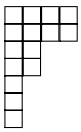
$$C_3^{(15)} = (I_2 I_3 = 0)_{[1^6\,1^5\,1^4]} =$$



$$C_4^{(15)} = \left(I_1^2 I_3 + I_1 I_2^2 + I_1 I_5 + I_2 I_3 = 0\right)_{[1^6\,2^3\,1^2\,1]} =$$

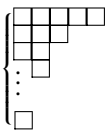


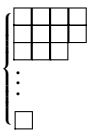
$$C_5^{(15)} = \left(I_2 I_3 + I_1 I_6 = 0\right)_{[1^7\,1^4\,2^2]} = \quad .$$

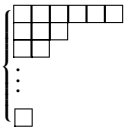


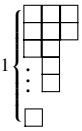
In these expressions for the constraints, we indicate that a linear combination of invariants is constrained, along with the Young tableau describing how the indices are meant to be contracted. We did not check whether some of the coefficients in these linear combinations could be zero. And in degree 18, there are nine more invariants:

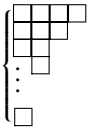
$$I_{12} = (\mathbf{27})^{18}_{[1^{10}\,1^4\,1^2\,2]} = \quad I_{13} = (\mathbf{27})^{18}_{[1^{10}\,2^3\,1^2]} =$$

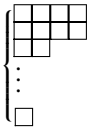


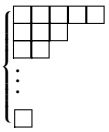


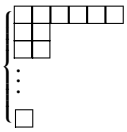
$I_{14} = (\mathbf{27})^{18}_{[1^{10}\,1^3\,1^2\,3]} =$ 

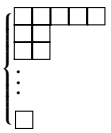
$I_{15} = (\mathbf{27})^{18}_{[1^{11}\,1^5\,1^2]} =$ 

$I_{16} = (\mathbf{27})^{18}_{[1^{11}\,1^4\,1^2\,1]} =$ 

$I_{17} = (\mathbf{27})^{18}_{[1^{11}\,1^3\,2^2]} =$ 

$I_{18} = (\mathbf{27})^{18}_{[1^{11}\,1^3\,1^2\,2]} =$ 

$I_{19} = (\mathbf{27})^{18}_{[1^{11}\,1^3\,4]} =$ 

$I_{20} = (\mathbf{27})^{18}_{[1^{12}\,1^3\,3]} =$  .

There are many constraints in degree 18, and we list them in an appendix. It is possible that this list of 20 invariants is not an exhaustive list of invariants. It would take over 50 days of computer time on a Pentium 4 machine to get the invariants of degree 21, using 1 GB of RAM, without improvement in the group theory software. We expect, however, that our list is almost complete. In fact, we seem to have all the invariants required for the subgroups of E_6. Along the $SO(10)$ flat direction for example, we ought to find a completely antisymmetrized baryon for the ten-dimensional vector: $(\mathbf{10}^{10}) = [1^{10}]$. This invariant of $SO(10)$ must come from the invariant I_{11} of E_6. As usual, information can be inferred about the invariants of the theories along the flat directions of E_6 from the knowledge of the basic invariants I_1 through I_{20}.

An understanding of the glueball invariants is also essential for the study of duality. This is particularly relevant for E_6 since it has $SO(N)$ subgroups, for which the glueballs play a crucial role in the matching of invariants [6]. Unfortunately, there are so many glueball invariants that we chose not to study the situation in more detail. Some results are to be found in the appendix.

4.1. Status of the self-dual model for E_6 with six flavours

In [52], Cho studied in detail the flat directions of the E_6 self-dual model of [48–50]. He considered the model along the electric Spin(8) flat direction, and found an invariant d'' of the E_6 dual theory with the quantum numbers $(\square, 5/2)$ under the global symmetry $SU(4) \times U(1)_R$. However, we point out that there are four invariants in the tensor product $8_v^3 \otimes 8_s \otimes 8_c$, four invariants in $8_s^3 \otimes 8_v \otimes 8_c$ and four invariants in $8_c^3 \otimes 8_v \otimes 8_s$ all with R-charge $5/2$. There is not a scarcity of invariants that d'' could correspond to, although a more detailed analysis would be required to tell which of these 12 (not all independent) invariants is the right one. Thus we do not see this problem as invalidating this duality.

A potentially more serious problem is the mapping of the previously unknown invariant I_3. Corresponding to I_3, there is an invariant

$\tilde{\mathcal{I}} =$.

However, $\tilde{\mathcal{I}}$ is not a basic invariant: it is simply the product of $\tilde{I}_1$ and $\tilde{I}_2$ from the dual theory. Furthermore, the dual superpotential renders $\tilde{I}_1$ redundant. Could $\tilde{\mathcal{I}}$ be considered to be a bona fide basic invariant?

Another puzzle is that the dual theory has an invariant $\tilde{I}_3$. This corresponds to

$\mathcal{I} =$

in the electric theory. $\mathcal{I}$ is certainly not a basic invariant, being the product of I_1 and I_2. Therefore, we need to get rid of $\tilde{I}_3$. Perhaps $\tilde{I}_3$ is made redundant by the dual superpotential. That might force us to relax the requirements of duality to rational or algebraic independence of the invariants instead of the stronger polynomial independence. Our attitude is that this self-dual model is complicated and that we do not have the technology to decisively confirm or disprove this duality[3].

5. Invariants of E_7 with fundamentals

We repeat the search for invariants of the previous section, but this time with the 56-dimensional fundamental representation of E_7. The well known invariants are

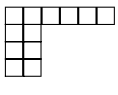
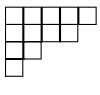
$I_1 = (\mathbf{56})^2_{[1^2]} =$ $I_2 = (\mathbf{56})^4_{[4]} =$

$I_3 = (\mathbf{56})^6_{[3^2]} =$ $I_4 = (\mathbf{56})^8_{[2^3\,2]} =$.

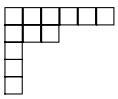
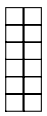
We then find three invariants of degree 10:

$I_5 = (\mathbf{56})^{10}_{[2^4\,1^2]} =$ $I_6 = (\mathbf{56})^{10}_{[1^4\,1^3\,3]} =$ $I_7 = (\mathbf{56})^{10}_{[1^4\,2^3]} =$.

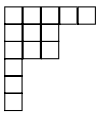
Nine invariants of degree 12:

$I_8 = (\mathbf{56})^{12}_{[4^3]} =$ $I_9 = (\mathbf{56})^{12}_{[2^4\,4]} =$  $I_{10} = (\mathbf{56})^{9}_{[1^4\,1^3\,2^2\,1]} =$ 

$I_{11} = (\mathbf{56})^{12}_{[2^5\,2]} =$ $I_{12} = (\mathbf{56})^{12}_{[1^5\,1^4\,1^2\,1]} =$ $I_{13} = (\mathbf{56})^{12}_{[1^5\,2^3\,1]} =$

$I_{14} = (\mathbf{56})^{12}_{[1^5\,1^3\,1^2\,2]} =$ $I_{15} = (\mathbf{56})^{12}_{[1^5\,2^2\,3]} =$  $I_{16} = (\mathbf{56})^{12}_{[2^6]} =$  .

We then find 30 invariants of degree 14, where we begin to find a multiplicity of invariants with identical transformation properties. For example, the computer counted a total of 14 invariants with the Young tableau shape:



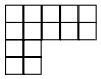
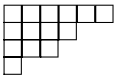
but then found that this Young tableau can be obtained in 12 different ways from the lower degree invariants[4], leaving us with a count of 2 new basic invariants with that particular shape.

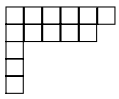
[3] There is another serious problem with this self-dual model [79], with the $\mathbb{Z}_{36}$ global symmetry. One way around this problem is to add the invariant I_2 to the superpotential of both electric and magnetic theories. The resulting global symmetry is $\mathrm{Spin}(6) \times U(1)_R \times \mathbb{Z}_6$. We thank Andreas Karch for reminding us.

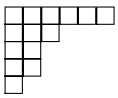
[4] Namely $I_1^5 I_2$, $I_1^4 I_3$, $I_1 I_3^2$, $I_1 I_{13}$, $I_1 I_{14}$, $2 \times I_1^2 I_2 I_3$, $2 \times I_1^2 I_6$, $3 \times I_1^3 I_4$.

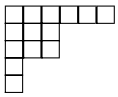
Thus the new invariants of degree 14 are

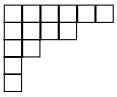
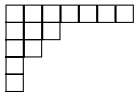
$(\mathbf{56})^{14}_{[3^4\,1^2]} =$ $(\mathbf{56})^{14}_{[2^4\,1^3\,1^2\,1]} =$

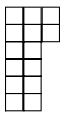
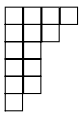
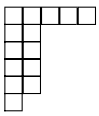
$(\mathbf{56})^{14}_{[2^4\,3^2]} =$  $(\mathbf{56})^{14}_{[1^4\,2^3\,1^2\,2]} =$ 

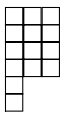
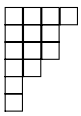
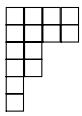
$(\mathbf{56})^{14}_{[2^5\,1^2\,2]} =$ $(\mathbf{56})^{14}_{[1^5\,2^4\,1]} =$  $(\mathbf{56})^{14}_{[1^5\,1^4\,1^3\,1^2]} =$

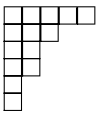
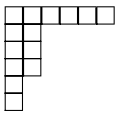
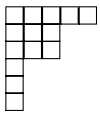
$(\mathbf{56})^{14}_{[1^5\,1^4\,1^3\,2]} =$ $(\mathbf{56})^{14}_{[1^5\,1^4\,2^2\,1]} =$ $(\mathbf{56})^{14}_{[1^5\,1^4\,1^2\,3]} =$ 

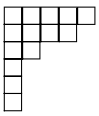
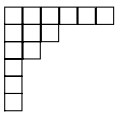
$(\mathbf{56})^{14}_{[1^5\,2^3\,1^2\,1]} =$ $(\mathbf{56})^{14}_{[1^5\,2^3\,3]} =$ 

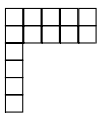
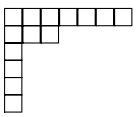
$(\mathbf{56})^{14}_{[1^5\,1^3\,2^2\,2]} =$  $(\mathbf{56})^{14}_{[1^5\,1^3\,1^2\,4]} =$ 

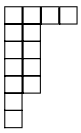
$(\mathbf{56})^{14}_{[2^6\,1^2]} =$  $(\mathbf{56})^{14}_{[1^6\,1^5\,1^2\,1]} =$  $(\mathbf{56})^{14}_{[1^6\,1^5\,3]} =$ 

$(\mathbf{56})^{14}_{[1^6\,2^4]} =$  $(\mathbf{56})^{14}_{[1^6\,1^4\,1^3\,1]} =$  $2\times\ (\mathbf{56})^{14}_{[1^6\,1^4\,2^2]} =$ 

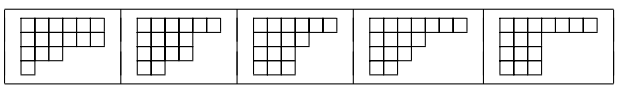
$(\mathbf{56})^{14}_{[1^6\,1^4\,1^2\,2]} =$  $(\mathbf{56})^{14}_{[1^6\,1^4\,4]} =$  $2\times\ (\mathbf{56})^{14}_{[1^6\,2^3\,2]} =$ 

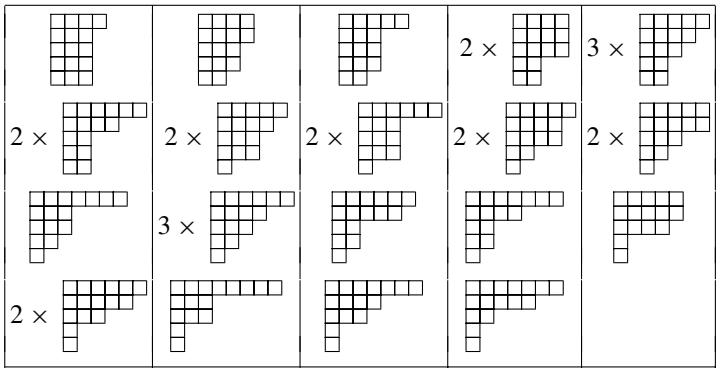
$(\mathbf{56})^{14}_{[1^6\,1^3\,2^2\,1]} =$  $(\mathbf{56})^{14}_{[1^6\,1^3\,1^2\,3]} =$ 

$(\mathbf{56})^{14}_{[1^6\,4^2]} =$  $(\mathbf{56})^{14}_{[1^6\,2^2\,4]} =$ 

$(\mathbf{56})^{14}_{[1^7\,1^5\,2]} =$  .

Then we find 125 basic invariants of degree 16 (from here on, we find the Young tableau generating macro written by Distler for [50] extremely useful):

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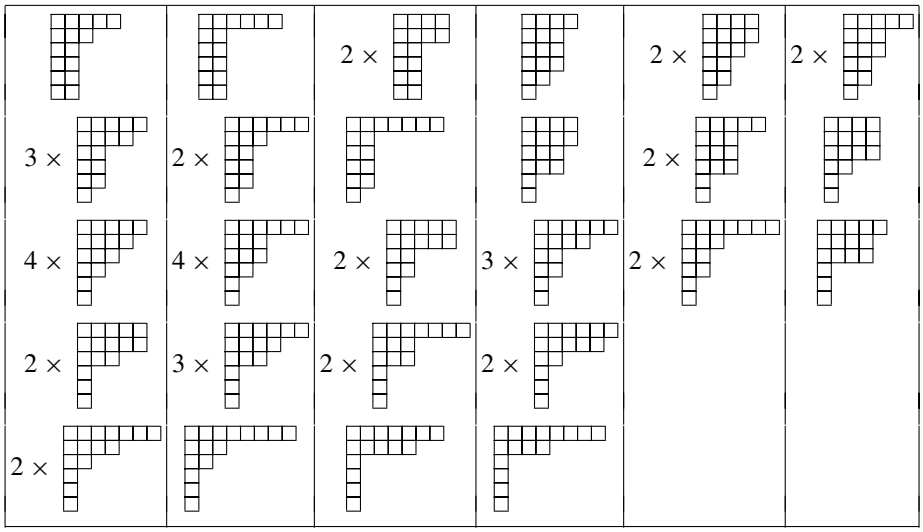
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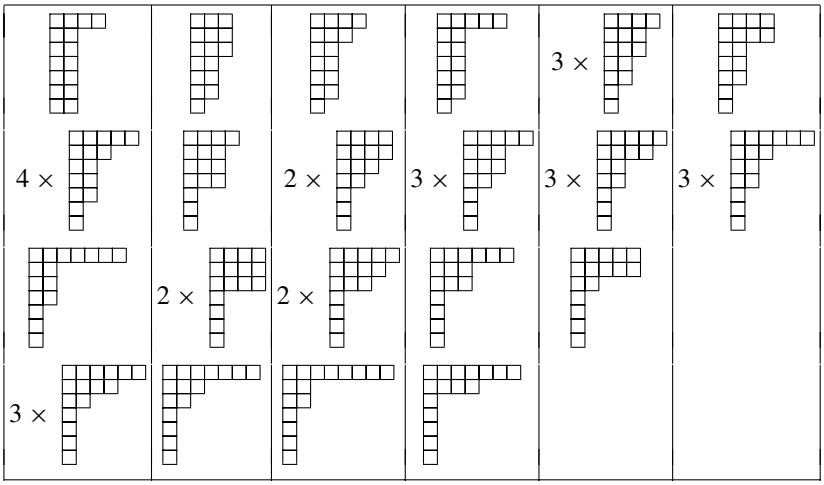
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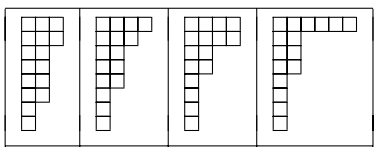
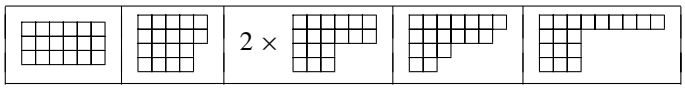
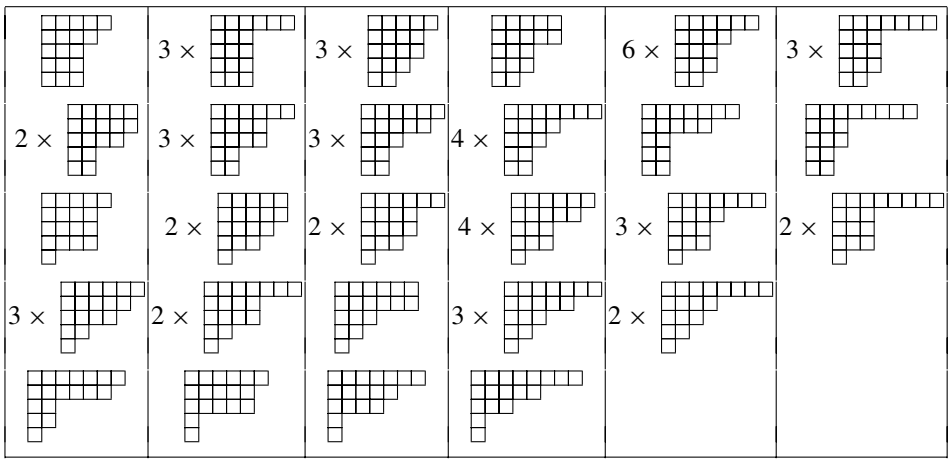
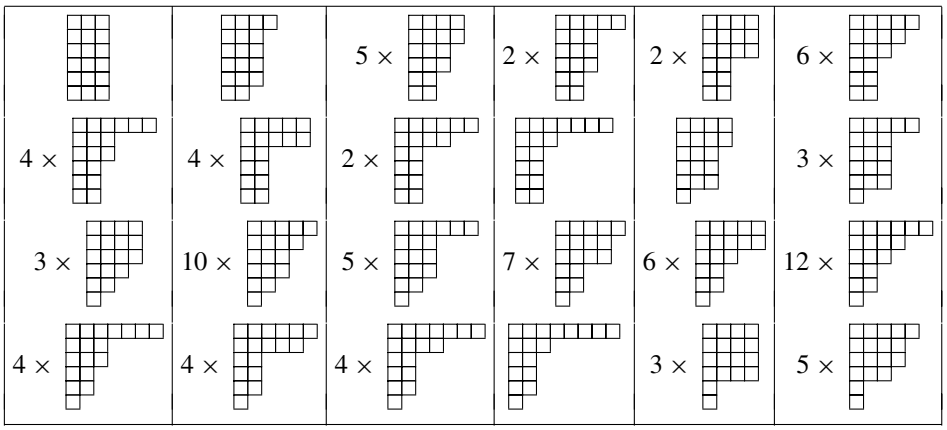
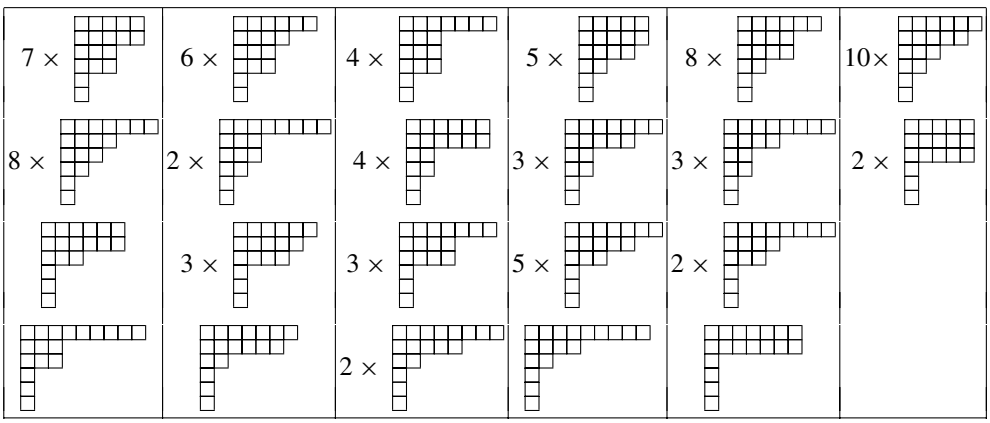


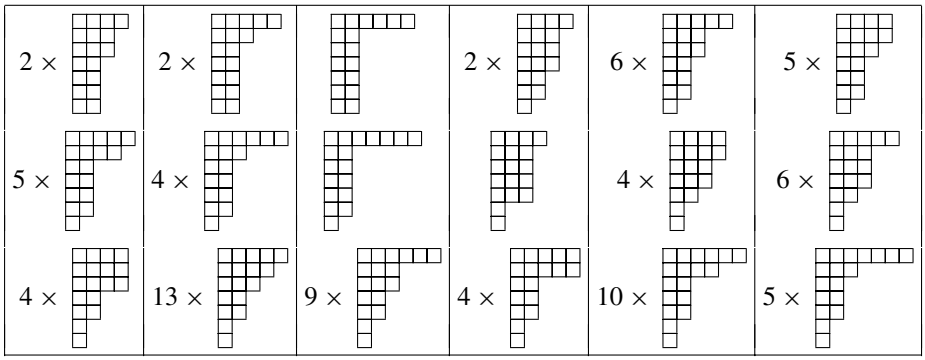
We now find 569 fundamental invariants of degree 18:







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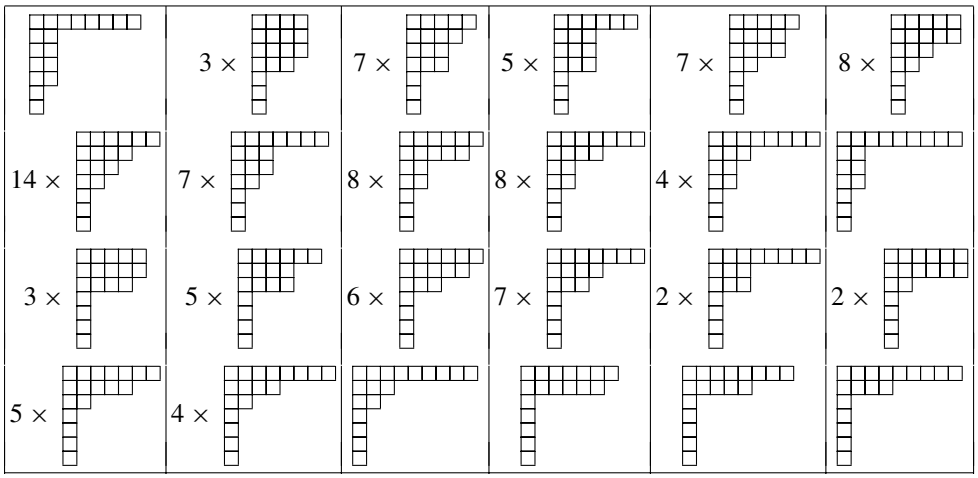
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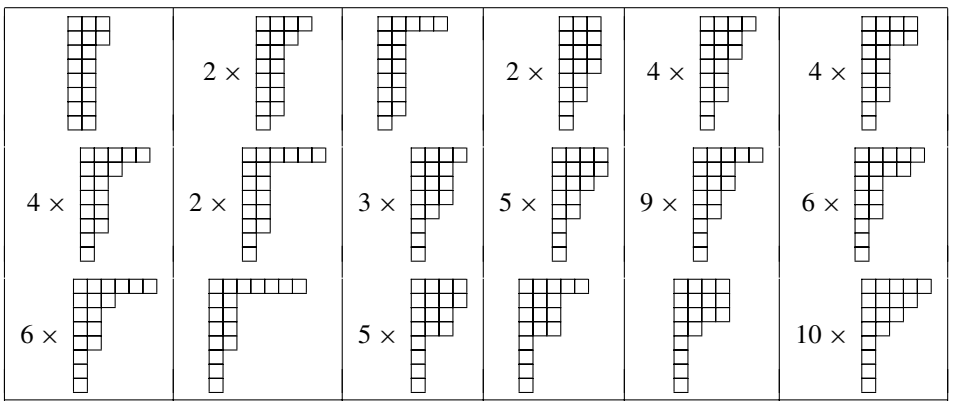
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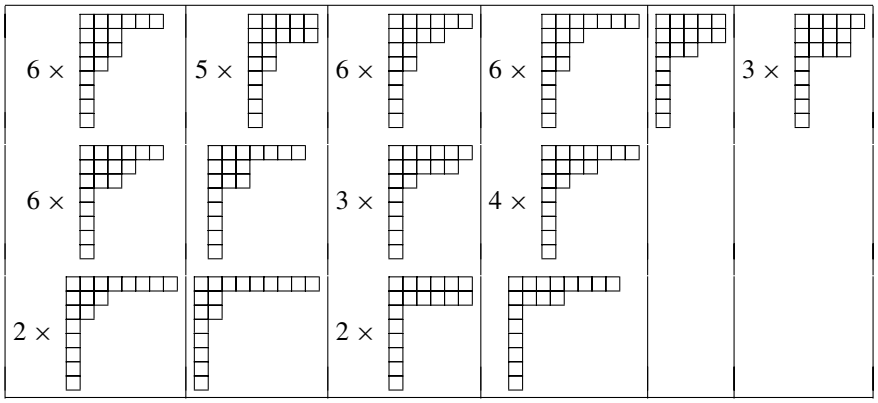
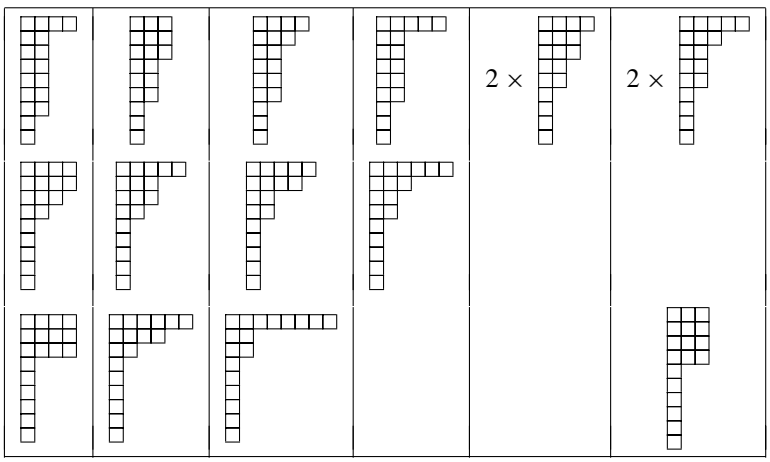
9 ×
6 ×
6 ×
5 ×
10 ×





So far, we have identified 740 basic invariants, up to degree 18. Since E_7 has an $SO(12)$ subgroup, there is at least a $(\mathbf{12})^{12}$ invariant along this flat direction. We found only one invariant with as many as ten antisymmetrized boxes thus far. This indicates that many more invariants are yet to be found.

5.1. Status of the self-dual model for E_7 with four flavours

In [52], Cho studied in detail the flat directions of the E_7 self-dual model of [48,50] and found some problems in matching invariants. As for E_6 with six flavours, we find that invariants had been missed; while some of the new invariants solve the problems mentioned in [52], others cause new problems. The list of invariants for E_7 with four flavours have the following

$SU(4) \times U(1)_R$ charge:

(, 1/2)	(, 1)	(, 3/2)
(, 2)	(, 5/2)	(, 5/2)
(, 5/2)	(, 3)	(, 3)
(, 3)	(, 7/2)	(, 7/2)
(, 7/2)	(, 7/2)	(, 4)
(, 4)	(, 4)	(, 4)
(, 4)	(, 9/2)	(, 9/2)
(, 9/2)	(, 9/2)	(, 9/2)

A few representations in this table are complex and are then potentially dangerous. A much more detailed analysis would be required to argue either way whether this duality is valid or not[5].

6. Examples with other groups

We have chosen a few other examples that illustrate the complexity that one encounters for some familiar rings that arise for supersymmetric gauge theories.

6.1. The ring of $SU(2)$ with fundamentals

In this section, we describe the simplest example of all in greater detail. We construct explicitly the free resolution of the rings, again up to degree 18. The moduli space of $SU(2)$ with $2N$ doublets is described by one flat direction. Alternatively, it can be described in a gauge-invariant way by the invariant [2]:

$$V_{ij} = \epsilon_{\alpha\beta} Q_i^\alpha Q_j^\beta = \square\!\!\square .$$

This statement is known as the 'first fundamental theorem of classical invariant theory', and has of course appropriate generalizations for the SU, SO and Sp groups. This invariant satisfies a single constraint (one first-order syzygy):

$$Z_1 = \text{Pf } V = \begin{array}{|c|}\hline\ \\\hline\ \\\hline\ \\\hline\end{array} .$$

That statement is known as the 'second fundamental theorem'. This constraint is itself constrained (one second-order syzygy):

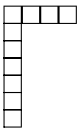
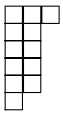
$$Z_2 = \begin{array}{|c|c|}\hline\ & \\\hline\ & \\\cline{1-1}\ & \\\cline{1-1}\ & \\\cline{1-1}\end{array} .$$

[5] Just as for the E_6 self-dual model, there is a serious problem with matching the global discrete symmetry. One way around this problem is to add the invariant I_4 to the superpotential of both electric and magnetic theories. This reduces the global $SU(4)$ symmetry to $SU(3)$.

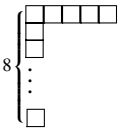
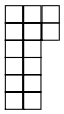
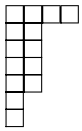
The second-order syzygy is constrained by two third-order syzygies:

$Z_3^1 =$ $Z_3^2 =$.

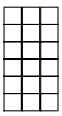
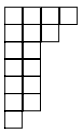
The third-order syzygies are constrained by two fourth-order syzygies:

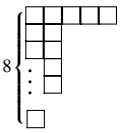
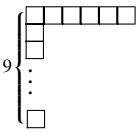
$Z_4^1 =$  $Z_4^2 =$  .

The fourth-order syzygies are constrained by three fifth-order syzygies:

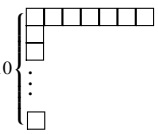
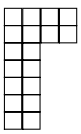
$Z_5^1 =$  $Z_5^2 =$  $Z_5^3 =$  .

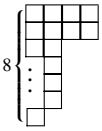
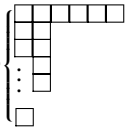
The fifth-order syzygies are constrained by four sixth-order syzygies:

$Z_6^1 =$  $Z_6^2 =$ 

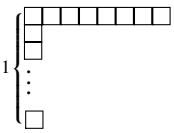
$Z_6^3 =$  $Z_6^4 =$  .

And the sixth-order syzygies are constrained by four seventh-order syzygies:

$Z_7^1 =$  $Z_7^2 =$ 

$Z_7^3 =$  $Z_7^4 =$  .

Then the seventh-order syzygies are constrained by at least one eighth-order syzygy:

$Z_8 =$  ,

and so on and so forth. Exhibiting this syzygy chain is known as constructing the free resolution of the ring. To our knowledge, the detailed form of higher-order syzygies has not played a role in physics.

6.2. Invariants and syzygies of G_2 with fundamentals

It is only since the 1980s [64, 65] that the first and second fundamental theorems have been proven for G_2, and of course the other exceptional groups are well beyond reach. In this

section, we will verify explicitly the results of [64, 65] for the invariants and the first-order syzygies using our computer-based method. We find a complete agreement, and we extend the results to the second-order syzygies. For the seven-dimensional fundamental representation, the invariants are well known to be

$$I_1 = (\mathbf{7})^2_{[2]} = \ydiagram{2} \qquad I_2 = (\mathbf{7})^3_{[1^3]} = \ydiagram{1,1,1} \qquad I_3 = (\mathbf{7})^4_{[1^4]} = \ydiagram{1,1,1,1}\,.$$

There are first-order constraints among these invariants:

$$C_{(6)} = \left(I_1 I_3 + I_2^2 = 0\right)_{[1^5\,1]} = \ydiagram{2,1,1,1,1} \qquad C^1_{(7)} = (I_2 I_3 = 0)_{[1^6\,1]} = \ydiagram{2,1,1,1,1,1}$$

$$C^2_{(7)} = \left(I_1^2 I_2 + I_2 I_3 = 0\right)_{[1^5\,1^2]} = \ydiagram{2,2,1,1} \qquad C^1_{(8)} = \left(I_3^2 = 0\right)_{[1^8]} = 8\left\{\ydiagram{1,1,1,1,1,1,1,1}\right.$$

$$C^2_{(8)} = \left(I_1^4 + I_3^2 + I_1 I_2^2 = 0\right)_{[2^4]} = \ydiagram{2,2,2,2} \qquad C^3_{(8)} = \left(I_1^2 I_3 + \alpha\, I_1 I_2^2 + I_3^2 = 0\right)_{[1^6\,1^2]} = \ydiagram{2,2,1,1,1,1}\,.$$

The subscript for C denotes the degree of the constraint, while the superscript just enumerates them. It is claimed in [65] that these are all the constraints, and our explicit calculation checks this to degree 16. The information in [65] is more detailed than what our computer-based method lets us achieve. For example, we see, from table I in [65], that the coefficient α is zero in the constraint $C^3_{(8)}$. Note also that our constraints are listed as (5.4.1) through (5.4.5) of table 1. The other constraints in table I, as shown in [65], follow from the constraints (5.4.1) through (5.4.5), and therefore should not be included as independent constraints; our calculation explicitly confirms that. We now move on beyond the results of [65]. In degree 9, we find two second-order syzygies, i.e. constraints amongst the first-order constraints:

$$Z^1_{(9)} = \left(I_1 C^1_{(7)} + I_1 C^2_{(7)} + I_2 C_{(6)} = 0\right)_{[1^6\,1^2\,1]} = \ydiagram{3,2,1,1,1}$$

$$Z^2_{(9)} = \left(I_1 C^1_{(7)} + I_2 C_{(6)} = 0\right)_{[1^7\,1^2]} = \ydiagram{2,2,1,1,1,1}\,.$$

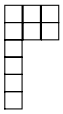
In degree 10, we find 12 second-order syzygies:

$$Z^1_{(10)} = \left(I_1 C^2_{(8)} + I_2 C^2_{(7)} + I_3 C_{(6)} = 0\right)_{[1^5\,1^4\,1]} = \ydiagram{3,2,2,1}$$

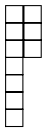
$$Z^2_{(10)} = \left(I_1 C^3_{(8)} + I_2 C^1_{(7)} + I_2 C^2_{(7)} + I_1^2 C_{(6)} + I_3 C_{(6)} = 0\right)_{[1^6\,1^3\,1]} = \ydiagram{3,2,1,1,1}$$

$$Z^{3,4}_{(10)} = \left(I_2 C^1_{(7)} + I_2 C^2_{(7)} + I_3 C_{(6)} = 0\right)_{[1^6\,1^4]} = \ydiagram{2,2,2,1,1}$$

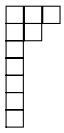
$$Z^{5}_{(10)} = \left(I_2 C^{2}_{(7)} + I_1^2 C_{(6)} = 0\right)_{[1^6\,2^2]} =$$



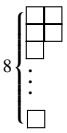
$$Z^{6}_{(10)} = \left(I_1 C^{3}_{(8)} + I_2 C^{1}_{(7)} + I_2 C^{2}_{(7)} + I_3 C_{(6)} + I_1^2 C_{(6)} = 0\right)_{[1^7\,1^3]} =$$



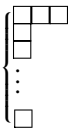
$$Z^{7,8}_{(10)} = \left(I_1 C^{3}_{(8)} + I_2 C^{1}_{(7)} + I_2 C^{2}_{(7)} + I_1^2 C_{(6)} + I_3 C_{(6)} = 0\right)_{[1^7\,1^2\,1]} =$$



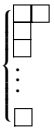
$$Z^{9,10}_{(10)} = \left(I_2 C^{1}_{(7)} + I_2 C^{2}_{(7)} + I_3 C_{(6)} = 0\right)_{[1^8\,1^2]} =$$



$$Z^{11}_{(10)} = \left(I_1 C^{1}_{(8)} + I_2 C^{1}_{(7)} + I_3 C_{(6)} = 0\right)_{[1^8\,2]} =$$

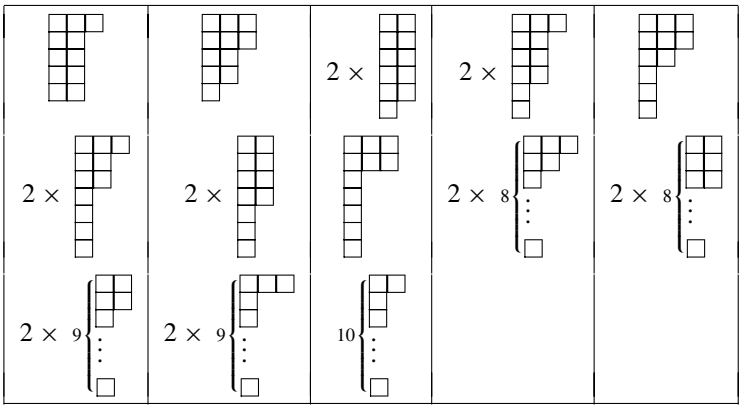


$$Z^{12}_{(10)} = \left(I_1 C^{1}_{(8)} + I_2 C^{1}_{(7)} + I_3 C_{(6)} = 0\right)_{[1^9\,1]} =$$

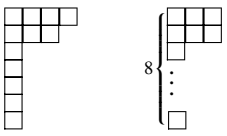


.

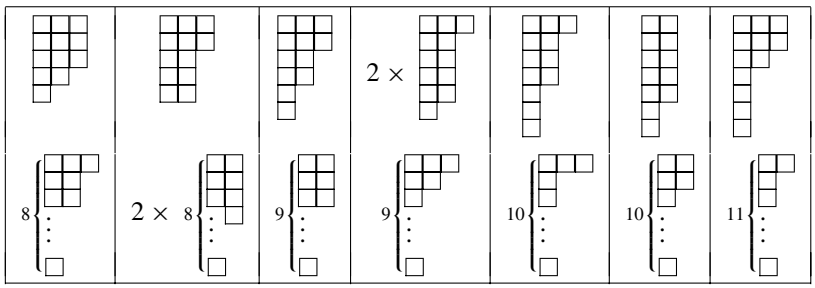
In degree 11, there are 21 second-order syzygies:



In degree 12, we find the first third-order syzygies:

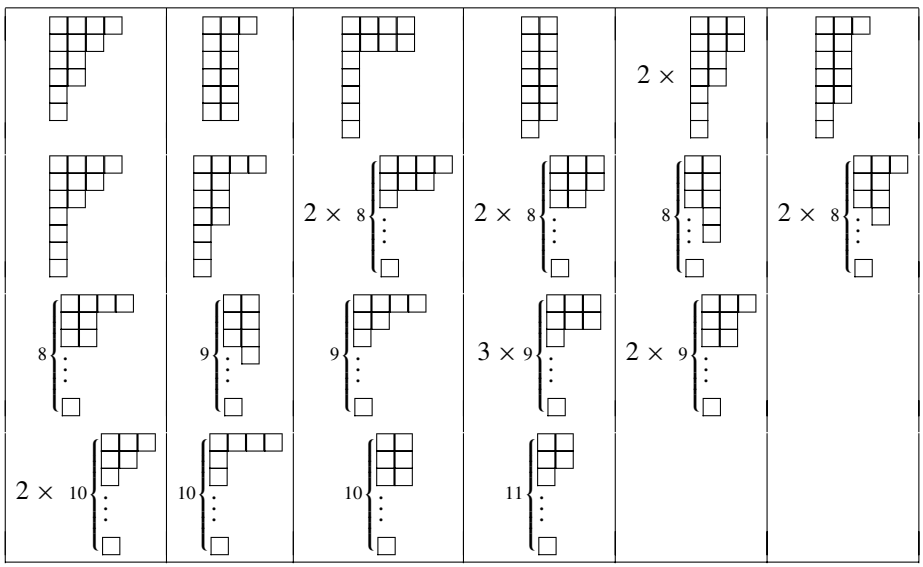


as well as 16 more second-order syzygies:



In degree 13, there is only one second-order syzygy:

but there are many third-order syzygies:



In degree 14, 15 and 16, we found no new second-order syzygy (but hundreds of new third-order syzygies).

6.3. Invariants of SU(3) *with symmetric tensors*

We repeat our search for the invariants of the six-dimensional symmetric tensor of $SU(3)$. We find this of interest because of the close resemblance, initially, between the chiral ring of the 6 of $SU(3)$ and that of the 27 of E_6. For the six-dimensional symmetric representation, the invariants are

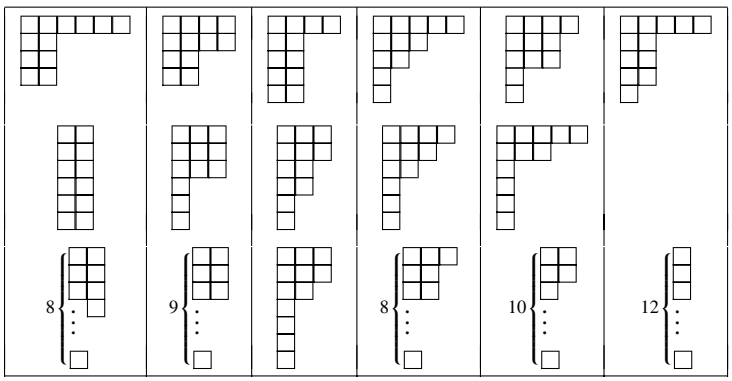
$$I_1 = (\mathbf{6})^3_{[3]} = \square\square\square \qquad I_2 = (\mathbf{6})^6_{[2^3]} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \end{array}$$

$$I_3 = Q^9_{[1^6 2^2]} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \hline \ \\ \cline{1-1} \end{array} \qquad I_4 = (\mathbf{6})^6_{[1^6]} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array}.$$

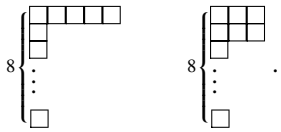
These appears to be all the invariants (at least up to degree 18). There is one constraint in degree 9:

$$C_{(9)} = (I_1 I_3 = 0)_{[1^7\,2]} =$$.

In degree 12, we find 17 constraints:



As well as degree 12 second-order syzygies:

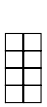


.

6.4. Invariants of SU(4) *with symmetric tensors*

For $SU(4)$ with a ten-dimensional symmetric tensor, there is one invariant of degree 4 and three of degree 8:

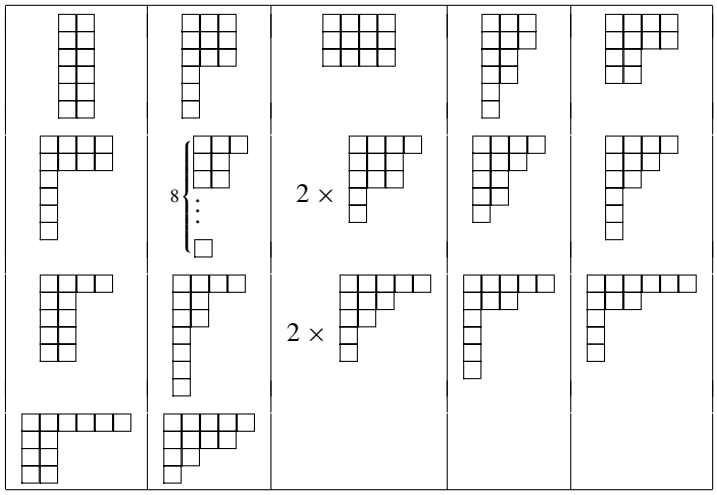
$$I_1 = (\mathbf{10})^4_{[4]} =$$

$$I_2 = (\mathbf{10})^8_{[2^4]} =$$ 

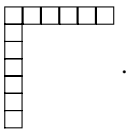
$$I_3 = (\mathbf{10})^8_{[1^6\,2]} =$$

$$I_4 = (\mathbf{10})^8_{[2^3\,2]} =$$.

We then find a large number of invariants of degree 12:



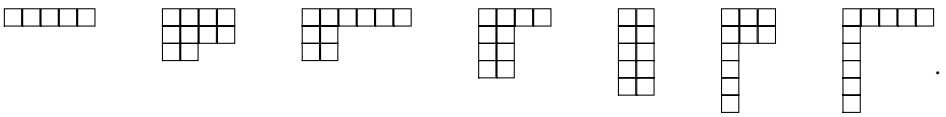
As well as one constraint of degree 12:



There is an impressionistic resemblance between the chiral ring of symmetric tensors of $SU(4)$ and the chiral ring of E_7 with fundamentals, at least for the invariants of degrees 4, 8 and 12.

6.5. Invariants of $SU(5)$ with symmetric tensors

For the 15-dimensional symmetric tensor representation of $SU(5)$, the first few invariants are



Then in degree 15, there is roughly 178 new invariants and one constraint.

7. Conclusion

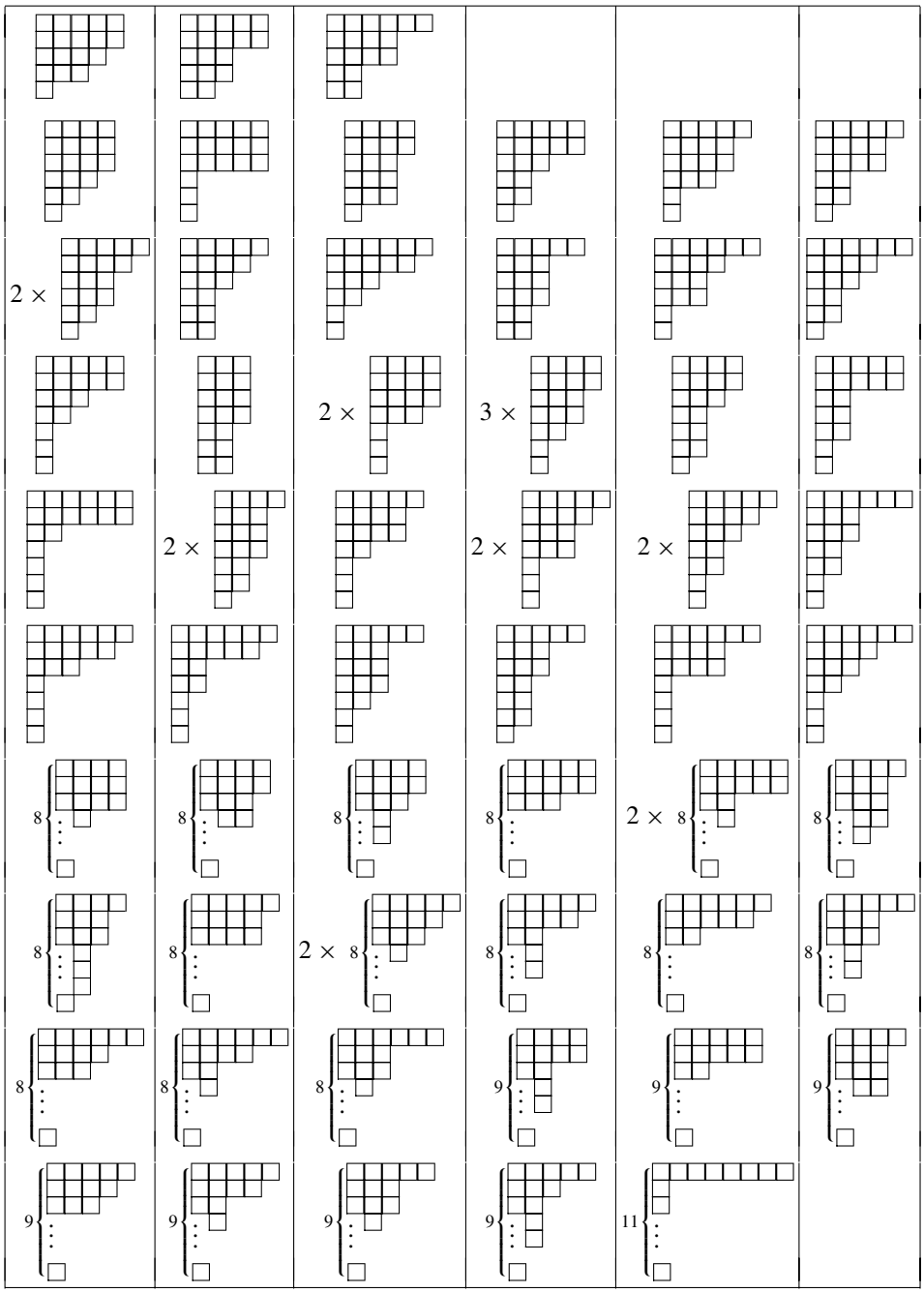
In this paper, we have explored the structure of the chiral rings of several classes of gauge theories. We were surprised that the invariants turned out to be so complicated. This makes the search for more duals at least arduous, if not futile.

There might be a guide for finding more gauge theories that have a chance of having a simple enough solution: one can systematically calculate the homological dimension of rings of gauge theories and look for examples, (e.g. by truncating the chiral ring [23]), which have a small homological dimension.

Acknowledgments

We would like to thank Dan Freed for explanations putting classical invariant theory in context. This work was supported by NSF grant PHY-0071512.

Appendix A. The constraints of degree 18 for E_6

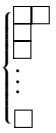


Appendix B. Some glueballs of E_6

In general, we do not understand how glueballs are mapped. For example, in the basic example of $SU(N_c) \leftrightarrow SU(N_f - N_c)$ of [1], the spinorial glueball superfields do not appear to have

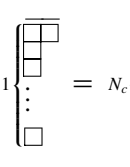
partners in the dual theory[6]:

$$W_\alpha Q^{N_c} = {}_{N_c-1}\left\{\begin{array}{l}\ydiagram{2,1,1,1}\\ \vdots\\ \ydiagram{1}\end{array}\right.$$



in the electric theory, clearly transforms differently from

$$\tilde{W}_\alpha q^{N_f-N_c} = {}_{N_f-N_c-1}\left\{\begin{array}{l}\overline{\ydiagram{2,1,1,1}}\\ \vdots\\ \ydiagram{1}\end{array}\right. = {}_{N_c-1}\left\{\begin{array}{l}\ydiagram{2,2,2,2}\\ \vdots\ \vdots\\ \ydiagram{2,1}\end{array}\right\}{}_{N_f-1}$$



in the magnetic theory.

For E_6, the situation is of course much worse. We find that there is a rather large number of glueballs for E_6, the invariants involving the 78-dimensional adjoint representation for the glueball superfield W_α. First, the Lorentz-spinor invariants of the form $W_\alpha Q^{3n}$:

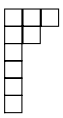
$J_1 = (\mathbf{78})\,(\mathbf{27})^3_{[1^2\,1]} = \ydiagram{2,1}$ $\qquad J_2 = (\mathbf{78})\,(\mathbf{27})^6_{[1^3\,1^2\,1]} = \ydiagram{3,2,1}$

$J_3 = (\mathbf{78})\,(\mathbf{27})^6_{[1^4\,1^2]} = \ydiagram{2,2,1,1}$ $\qquad J_4 = (\mathbf{78})\,(\mathbf{27})^9_{[1^4\,2^2\,1]} = \ydiagram{4,2,1,1}$

$J_5 = (\mathbf{78})\,(\mathbf{27})^9_{[1^4\,1^3\,1^2]} = \ydiagram{3,2,2,1}$ $\qquad J_6 = (\mathbf{78})\,(\mathbf{27})^9_{[1^5\,2^2]} = \ydiagram{3,2,1,1,1}$

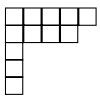
$J_7 = (\mathbf{78})\,(\mathbf{27})^9_{[1^5\,1^3\,1]} = \ydiagram{4,2,2,1,1}$ $\qquad J_8 = (\mathbf{78})\,(\mathbf{27})^9_{[1^5\,1^2\,2]} = \ydiagram{4,2,1,1,1}$

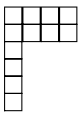
$J_9 = (\mathbf{78})\,(\mathbf{27})^9_{[1^6\,1^2\,1]} = \ydiagram{3,2,1,1,1,1}$



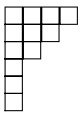
$J_{10} = (\mathbf{78})\,(\mathbf{27})^{12}_{[2^4\,1^3\,1]} = \ydiagram{4,4,2,1}$ $\qquad J_{11} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^5\,1^4\,1^3]} = \ydiagram{4,4,3,2,1}$

$J_{12} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^5\,1^3\,2^2]} = \ydiagram{4,3,2,1,1}$ $\qquad J_{13} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^5\,3^2\,1]} = \ydiagram{5,3,1,1,1}$

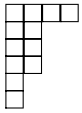


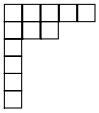


$J_{14} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^6\,3^2]} = \ydiagram{4,4,1,1,1,1}$ $\qquad J_{15,16} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^6\,1^3\,1^2\,1]} = \ydiagram{4,3,2,1,1,1}$

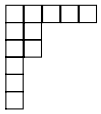


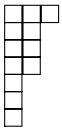
$J_{17} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^6\,1^4\,2]} = \ydiagram{4,2,2,1,1,1}$ $\qquad J_{18} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^6\,2^2\,2]} = \ydiagram{5,2,1,1,1,1}$

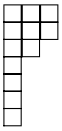


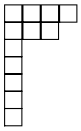


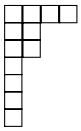
[6] This is actually not a problem because these invariants, although chiral, are not primary: they are descendants [81]. We will not try to discriminate between primaries and descendants in this appendix. We thank Andreas Karch and Micha Berkooz for this comment.

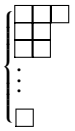
$J_{19} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^6\,1^3\,3]} =$ 

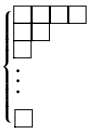
$J_{20} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^7\,1^4\,1]} =$ 

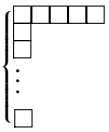
$J_{21} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^7\,1^3\,1^2]} =$ 

$J_{22,23,24} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^7\,1^2\,3]} =$ 

$J_{25,26} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^7\,1^3\,2]} =$ 

$J_{27} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^8\,1^3\,1]} =$ 

$J_{28} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^8\,1^2\,2]} =$ 

$J_{29} = (\mathbf{78})\,(\mathbf{27})^{12}_{[1^8\,4]} =$ 

.

Then the Lorentz-scalar invariants of the form $W_\alpha^2 Q^{3n}$:

$K_1 = (\mathbf{78})^2\,(\mathbf{27})^3_{[3]} =$

$K_2 = (\mathbf{78})^2\,(\mathbf{27})^3_{[1^2\,1]} =$

$K_3 = (\mathbf{78})^2\,(\mathbf{27})^3_{[1^2\,1]} =$

$K_4 = (\mathbf{78})^2\,(\mathbf{27})^3_{[1^3]} =$

$K_{5,6} = (\mathbf{78})^2\,(\mathbf{27})^6_{[2^3]} =$

$K_{7,8} = (\mathbf{78})^2\,(\mathbf{27})^6_{[3^2]} =$

$K_9 = (\mathbf{78})^2\,(\mathbf{27})^6_{[2^2\,2]} =$

$K_{10,11,12,13} = (\mathbf{78})^2\,(\mathbf{27})^6_{[1^4\,1^2]} =$

$K_{14,15,16,17,18} = (\mathbf{78})^2\,(\mathbf{27})^6_{[1^3\,1^2\,1]} =$

$K_{19,20} = (\mathbf{78})^2\,(\mathbf{27})^6_{[1^4\,2]} =$

$K_{21} = (\mathbf{78})^2\,(\mathbf{27})^6_{[1^5\,1]} =$

$K_{22,23} = (\mathbf{78})^2\,(\mathbf{27})^6_{[1^3\,3]} =$

.

Due to this proliferation of glueball invariants, we stop the list here.

Appendix C. Computer program code for LiE

The group theory program LiE can be obtained for free from a variety of sources, e.g. `http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/`. It is written in *C*, easy to install and easy to use. It comes with a well-written and useful manual [82]. (There is another well known group theory software, Schur. However, it is not free.) We include some commented code below.

```
0 #Write to a default file named 'monfile'
1 on monitor
2 maxobjects 2000000
3 maxnodes 12000
4 thegroup = E6; therank = 6;
5 #n is the degree of the invariants we are computing
6 n=9;
7 setdefault Lie_group(1,n);
```

```
8 fund = [1,0,0,0,0,0,0,0,0];
9 y= poly_null(n)
10 ii=0;
11 #The next line iterates over all the partitions of n=6
12 for lambda row partitions(n) do
13 #Compute the plethysm of degree n of E6
14 b=plethysm(lambda,[1,0,0,0,0,0],thegroup);
15 #Compute the SU(n) representation for the partition λ
16 x=plethysm(lambda,fund);
# Line 16 is very slow. More efficient is to use the built-in function from_part
# (after adding a zero to each partition to get representations of An instead of An−1). 17 #Print
useful information at each iteration
18 ii=ii+1;
19 print("partition number:"); print(ii);
20 print("partition:"); print(x); print(lambda);
21 print("number of objects used:"); print(used);
22 print("the plethysm:");
23 print(b);
24 print("Multiplying by W_alpha:");
25 #Tensor the plethysm b with the adjoint
26 #The coefficient 1X is for compatibility with b because b is a polynomial
27 cc=tensor(1X[0,1,0,0,0,0],b,thegroup);
28 print(cc);
29 #Pick the singlet(s) from the final expression
30 a=cc[1];
31 #Check that a is a singlet; if so, add it to y
32 if (a==(coef(a,1)* poly_one(therank))) then
33 y= y+( coef(a,1)*x);
34 print("This is the coefficient:");
35 print(coef(a,1)*x);
36 print("This is 'y':  "); print(y);
37 # 'y' is the full list of invariants
38 #Closes the if
39 fi;
40 #Activate the garbage collection:
41 #Essential memory management!
42 gcol;
43 #Close the do loop
44 od;
45 #Define the previously found lower degree invariants
46 j1 = plethysm([2,1],fund);
47 j2 = plethysm([3,2,1],fund);
48 j3 = plethysm([2,2,1,1],fund);
49
50 r1 = plethysm([3],fund);
51 r2 = plethysm([2,2,2],fund);
52 r3 = plethysm([3,3,1,1,1,1],fund);
53 r4 = plethysm([3,3,3,3],fund);
54 r5 = plethysm([4,4,1,1,1,1],fund);
```

```
55 r6 = plethysm([4,2,2,1,1,1,1],fund);
56 #Compute the product of lower degree invariants among themselves
57 f = tensor(r1,j2+j3)+tensor(sym_tensor(2,r1),j1)+tensor(r2,j1);
58 print("This is f:  ");
59 print(f);
60 print("These are the new fundamental invariants:  ");
61 print(y-f);
```

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